Unusual dynamical scaling in the spatial distribution of persistent sites in one-dimensional Potts models

A. J. Bray and S. J. O'Donoghue
Department of Physics and Astronomy, The University of Manchester, Manchester M13 9PL, United Kingdom
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The distribution $n(k,t)$ of the interval sizes k between clusters of persistent sites in the dynamical evolution of the one-dimensional q-state Potts model is studied using a combination of numerical simulations, scaling arguments, and exact analysis. It is shown to have the scaling form $n(k,t)=t^{-2z}f(k/t^z)$, with $z=\max(1/2,\theta)$, where $\theta(q)$ is the persistence exponent which describes the fraction $P(t)\sim t^{-\theta}$ of sites which have not changed their state up to time t. When $\theta>1/2$, the scaling length t^θ for the interval-size distribution is larger than the coarsening length scale $t^{1/2}$ that characterizes spatial correlations of the Potts variables.

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I. INTRODUCTION

The discovery of persistence has recently generated considerable interest in understanding the statistics of first-passage problems in spatially extended nonequilibrium systems, both theoretically and experimentally. The definition of persistence is as follows. Let $\phi(x,t)$ be a stochastic variable fluctuating in space and time according to some dynamics. The persistence probability is simply the probability $P(t)$ that at a fixed point in space, the quantity $\mathrm{sgn}[\phi(x,t)-\langle\phi(x,t)\rangle]$ does not change up to time t. In many systems of physical interest a power law decay $P(t)\sim t^{-\theta}$ is observed, where θ is the persistence exponent and is, in general, nontrivial. The nontriviality of θ emerges as a consequence of the coupling of the field $\phi(x,t)$ to its neighbors, since such coupling implies that the stochastic process at a fixed point in space and time is non-Markovian.

Persistence phenomena have been widely studied in recent years [1–12]. Theoretical and computational studies include spin systems in one [2,3] and higher [4] dimensions, diffusion fields [5], fluctuating interfaces [6], phase-ordering dynamics [7], and reaction-diffusion systems [8]. Experimental studies include the coarsening dynamics of breath figures [9], soap froths [10], and twisted nematic liquid crystals [11]. Persistence in nonequilibrium critical phenomena has also been studied in the context of the global order parameter $M(t)$ (e.g., the total magnetization of a ferromagnet), regarded as a stochastic process [12].

In the present work we consider spatially extended systems with a nonequilibrium field $\phi(x,t)$, which takes discrete values, at each lattice site x. The field then evolves in time t through interactions with its neighbors. The persistence probability at time t is defined as the fraction of sites in which the stochastic field $\phi(x,t)$ did not change its value in the time interval $[0,t]$. The physical interpretation of $\phi(x,t)$ could be, for example, the coarsening spin field in the Ising model after being quenched to low temperature from an initial high temperature, the sign of a diffusion field starting from a random initial configuration, or the sign of the height, relative to the mean height, of a fluctuating interface. As the stochastic field evolves in time, such systems develop regions of persistent and nonpersistent sites. Since the number of persistent sites decays with time according to $P(t)\sim t^{-\theta}$, the persistent clusters shrink in size and number with a corresponding growth in the size of the nonpersistent regions.

Recently, Manoj and Ray (MR) [13] have studied such spatially extended systems in one dimension (1D) within the context of the $A+A\to 0$ reaction diffusion model, which is equivalent to the 1D Ising model. They found that the length scale which characterizes the interval sizes between persistent clusters *apparently* has a different time dependence from the length scale characterizing the walker separations (or spin correlations, in the Ising representation), and futhermore seems to depend on the initial walker density. If true, this result would be surprising. Naively, one would expect the initial walker density to be irrelevant to the asymptotic dynamics and the coarsening length scale, set by the mean distance between walkers, to be the relevant length scale for all spatial correlations. We shall show that the former expectation is correct, but the latter, in general, is not.

In this paper we generalize, and reinterpret, this study in the context of the q-state Potts model, which has q distinct but equivalent ordered phases. The $A+A\to 0$ model corresponds to the $q=2$ Potts model, i.e., the Ising model, with the walkers identified as the domain walls between up and down Ising spins. In the q-state Potts model the random walkers represent domain walls between Potts states. At each time step, every walker hops randomly left or right. Persistent Potts sites are those at which the Potts state has never changed, i.e., sites which have never been crossed by a walker. If two walkers occupy the same site, they either *coalesce*, with probability $(q-2)/(q-1)$, or *annihilate*, with probability $1/(q-1)$, these numbers being the probabilities that the states on the farther sides of the walkers are the same (annihilation) or different (coalescence). The persistence probability (the fraction of sites that have never been jumped over by a walker) decays as a power of time, with a q-dependent exponent, $P(t)\sim t^{-\theta(q)}$.

Among the questions which emerge naturally in such a study are the following: (i) What is the dominant length scale in the problem, as far as the persistent structures are concerned? (ii) What is the nature of the spatial distribution of the persistent sites? (iii) What is the average size of a persistent cluster?

An important point to make at the outset concerns the different length scales associated with the walkers and with the persistent sites. The walker density decays as $t^{-1/2}$ for any q, so the mean distance between walkers grows as $L_w(t) \sim t^{1/2}$. The density of persistent sites decays as $t^{-\theta}$, so the mean distance between these sites grows as $L_p(t) \sim t^{\theta}$. This simple observation immediately suggests that the spatial structure of the persistent sites for $\theta > 1/2$, where L_p is the larger length scale, will be very different from when $\theta < 1/2$, where L_w is the larger length. This is precisely what we find: the characteristic length scale L_{int}, controlling the distribution of the *intervals* between clusters of persistent sites, is given by $L_{int} = \max(L_w, L_p)$. For the Ising case studied earlier [13], one has [3] $\theta = 3/8$, which is ''close'' to 1/2. We believe that the proximity of these two exponents is responsible for the apparently nonuniversal behavior in the Ising system reported in MR.

Derrida *et al.* [3] have obtained an exact expression for θ for 1D Potts models with arbitrary q:

$$\theta(q) = -\frac{1}{8} + \frac{2}{\pi^2}\left[\cos^{-1}\left(\frac{2-q}{\sqrt{2}q}\right)\right]^2. \tag{1}$$

The value of q corresponding to $\theta = 1/2$ is $q_c = 2/[1 + \sqrt{2}\cos(\sqrt{5}\pi/4)] = 2.705\,28\ldots$, so $\theta(q) > 1/2$ for all integer $q \geq 3$. Note that the probabilistic algorithm for implementing the Potts model through the annihilation or coalescence of random walkers (domain walls) allows q to be a real number, $q \geq 2$, while an equivalent Ising spin representation (see Sec. IV) of the Potts persistence problem even allows $1 < q < 2$! By this means we can explore a range of θ above and below 1/2.

Our main result is that the scale length controlling the distribution of interval sizes between persistent clusters is given by $t^{1/2}$ when $\theta < 1/2$, but by t^{θ} when $\theta > 1/2$. We find no evidence for any dependence of the asymptotic scaling distribution on the initial walker density (other than through nonuniversal amplitudes).

This paper is organized in the following manner. In Sec. II we outline a scaling phenomenology within which the above questions can be addressed. In Sec. III we give exact results for the $q = \infty$ Potts model. Finally, in Sec. IV, we present extensive numerical simulations for the q-state Potts model which confirm our predictions based on general scaling arguments and the exact large-q results. Section V concludes with a discussion and summary of the results.

II. SCALING PHENOMENOLOGY

The basic scaling phenomenology was introduced by Manoj and Ray, although we will adopt a slightly different notation. We are interested in the distribution of the sizes of the *intervals* between persistent sites. We define an interval size k as the number of *nonpersistent* sites between two consecutive persistent sites. If the interval size is zero, the sites belong to the same *cluster* of persistent sites, and we henceforth consider only intervals of nonzero size. We define $n(k,t)$ to be the number of intervals (per site) of size k at time t. A natural dynamical scaling assumption is

$$n(k,t) = t^{-2z} f(k/t^z), \tag{2}$$

where $L(t) = t^z$ is the ''characteristic'' length scale at time t. [A dynamical exponent z is conventionally defined via $L(t) \sim t^{1/z}$, rather than t^z. Here we are following the notation of MR.]

The rationale for the scaling form (2) is as follows. The number of nonpersistent sites (per site) is given by

$$Q(t) = \sum_{k=1}^{\infty} k n(k,t). \tag{3}$$

But since the number of persistent sites (per site), $P(t)$, decays to zero as $P(t) \sim t^{-\theta}$, and $P(t) + Q(t) = 1$, it follows that $Q(t) \to 1$ for $t \to \infty$. Converting the sum (3) to an integral, with a lower limit zero (valid as $t \to \infty$ provided the integral converges), we see that the prefactor t^{-2z} in Eq. (2) is precisely what is needed to satisfy the required condition $Q(t) \to$ const for $t \to \infty$.

Consider next the number $N_c(t)$ of persistent clusters per site. Since the number of clusters is equal to the number of intervals, N_c is given by

$$N_c(t) = \sum_{k=1}^{\infty} n(k,t). \tag{4}$$

Converting the sum to an integral (with a lower limit zero), using Eq. (2), and assuming the integral converges gives $N_c \sim t^{-z}$. Thus the mean distance between persistent clusters, i.e., the mean interval size, increases as t^z.

Is this reasonable? Let us recall that there are two length scales in the system, $L_w \sim t^{1/2}$, the mean distance between walkers, and $L_p \sim t^{\theta}$, the mean distance between persistent sites. To make further progress we make the following two assumptions, both of which are confirmed by our numerical studies, by exact results for $q = \infty$, and by heuristic arguments to be expounded below: (i) The mean cluster size tends to a constant for $t \to \infty$, and (ii) the length scale $L_{int} = t^z$ that controls the interval size distribution is the *larger* of L_w and L_p, i.e., $z = \max(1/2, \theta)$.

From assumption (i) we deduce that $N_c \sim t^{-\theta}$, i.e., $z = \theta$, which is consistent with assumption (ii) provided $\theta > 1/2$. What if $\theta < 1/2$? Then we still require $N_c \sim t^{-\theta}$, but now $z = 1/2$, so the result $N_c \sim t^{-z}$, derived from Eq. (4), breaks down. Going back to Eq. (4), we infer that this breakdown requires that the conversion of the sum to an integral be invalid—the sum must have its dominant contribution from k of order unity, rather than k of order t^z. This in turn requires that the scaling function $f(x)$ in Eq. (2) have a singular small-x limit of the form [13]

$$f(x) \sim x^{-\tau}, \quad x \to 0, \tag{5}$$

with $1 < \tau < 2$. Using this form in Eq. (2), with $z = 1/2$, and inserting the result into Eq. (4), gives $N_c \sim t^{-(1-\tau/2)}$. Comparing this with $N_c \sim t^{-\theta}$ fixes the value of τ:

$$\tau = 2(1-\theta), \quad \theta < 1/2. \tag{6}$$

We first present a heuristic argument for the result $z = \max(1/2, \theta)$. Consider first the case $\theta > 1/2$. In this regime, the distance between clusters is much greater that the distance between walkers; i.e., there are, on average, many

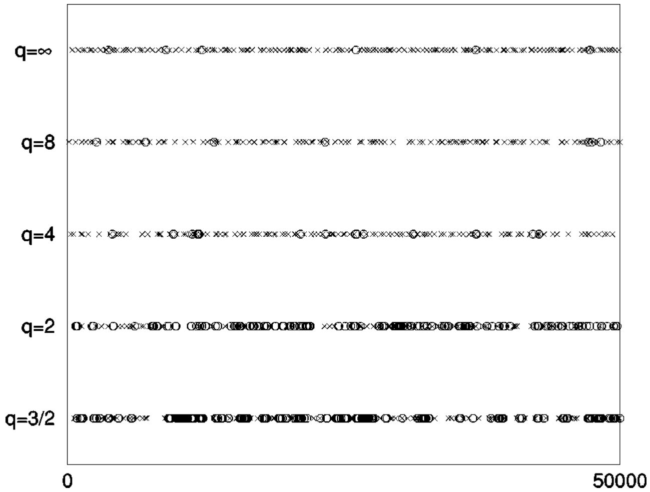

FIG. 1. Snapshots showing the relative densities of persistent sites (0) and random walkers (×) for various q, at time $t=10^4$. The Ising representation was used to generate the $q=3/2$ data, with the Potts representation used for the other q values.

($\sim t^{\theta-1/2}$) random walkers between each consecutive pair of persistent clusters, as illustrated in Fig. 1. However, walkers separated by distances large compared to $t^{1/2}$ are essentially uncorrelated, because any correlations are mediated by the random walkers, and correlations between walkers decay on the length scale $t^{1/2}$. Therefore one expects the intervals between clusters, measured on the scale t^{θ}, to be independent random variables and the interval size distribution to scale with this length. This argument also suggests that the locations of the clusters as a function of position on the lattice are described by a Poisson process; i.e., the interval size distribution is an exponential, $n(k,t)=\langle k\rangle\exp(-k/\langle k\rangle)$, with $\langle k\rangle\sim t^{\theta}$, for $\theta>1/2$. These expectations are borne out by our numerical studies.

For $\theta<1/2$, the opposite is true: there are many clusters between each pair of walkers (see the $q=3/2$ snapshot in Fig. 1). The largest length scale is set by the walker spacing, $L_w\sim t^{1/2}$, and the interval size distribution will scale with this length. On smaller scales, there will be residual structure in the cluster distribution left from an earlier epoch when the number of walkers was larger. Indeed, on these smaller scales the persistent sites form a fractal set [14–16]. Consider two sites separated by a distance r. The probability $P_2(r,t)$ that *both* are persistent has the scaling form

$$P_2(r,t)=t^{-2\theta}F(r/t^{1/2}), \tag{7}$$

where $F(x)\to$const for $x\to\infty$, since for $r\gg t^{1/2}$ the sites are uncorrelated. The residual structure remaining on scales $r\ll t^{1/2}$, however, means that if the first site is persistent, which occurs with probability of order $t^{-\theta}$, the probability that the second one is also persistent depends only on r. This implies $F(x)\sim x^{-2\theta}$ for $x\to 0$. The number of persistent sites within a distance R of a given persistent site can therefore be estimated as $\int_0^R dr\, r^{-2\theta}\sim R^{d_f}$, where

$$d_f=1-2\theta \tag{8}$$

is the fractal dimension of the persistent set.

Clearly this result only makes sense for $\theta<1/2$, since d_f cannot be negative. This suggests $d_f=0$ for $\theta\geq 1/2$, corresponding to pointlike objects, i.e., isolated finite clusters. For $q=\infty$, for example, where $\theta=1$, we find that there is typically one cluster of persistent sites, with a fixed mean size per scale length $L_p\sim t$.

We now present a heuristic argument in support of our claim that the mean cluster size, $\langle l\rangle$ approaches a constant as $t\to\infty$ for all values of θ. The initial steps of the argument follow the approach of MR. The total number of clusters per site can be written as

$$N_c(t)=P(t)-P_w(t), \tag{9}$$

where $P_w(t)$ is the fraction of walker sites which have never been visited by a walker, i.e., the persistence probability for walker sites. Such sites form ''spacers'' between each pair of persistent Potts states in a cluster, there being exactly one more persistent Potts site than persistent walker site in each cluster. The mean cluster size is

$$\langle l\rangle=\frac{P(t)}{N_c(t)}=\left[1-\frac{P_w(t)}{P(t)}\right]^{-1}. \tag{10}$$

We expect, on universality grounds, that the exponents describing the decay of $P(t)$ and $P_w(t)$ should be the same (i.e., θ), but the (nonuniversal) amplitudes will be different, i.e., $P(t)\to At^{-\theta}$, while $P_w(t)\to A_w t^{-\theta}$, with $A_w<A$. Inserting these forms in Eq. (10) gives the limiting value of the mean cluster size as $\langle l\rangle_\infty=(1-A_w/A)^{-1}$. This number is nonuniversal and is determined by the initial distribution of cluster sizes.

III. EXACT RESULTS FOR $A+A\to A$

Many of the general features of the dynamics for $\theta>1/2$ are exemplified by the $q\to\infty$ limit, for which the walker dynamics reduces to $A+A\to A$; i.e., the walkers always aggregate on contact. At $t=0$ all sites are persistent. The random walkers initially present divide the sites into clusters of persistent sites. Clearly no cluster can increase in size. Consider a cluster of initial size l_0. We first calculate the probability density $p_{l_0}(l,t)$ that the cluster survives to time t and has size l. The key point is that, for $A+A\to A$ dynamics, we need only consider the two walkers on either side of the initial cluster, since subsequent coalescence processes do not affect the random walk dynamics of these two walkers. The cluster and the two walkers can therefore be treated in isolation.

For simplicity we begin by treating the continuum limit in which the walkers are regarded as continuous-time random walkers on a continuous space. This should be correct in the limit $l_0\gg 1$, an assertion we can subsequently test. Let the ends of the initial cluster (and the two walkers) be located at $x=0$ and $x=l_0$. Each walker obeys an equation of the form $dx/dt=\xi(t)$, where $\xi(t)$ is a Gaussian white noise with mean zero and correlator $\langle\xi(t)\xi(t')\rangle=2\delta(t-t')$. We first write down the probability distribution $P_r(x_r,t)$ of the rightmost excursion x_r, up to time t, of the left walker. An elementary calculation gives

$$P_r(x_r,t)=\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{x_r^2}{4t}\right). \quad (11)$$

For the cluster to survive, we require $x_r<l_0$, so in the limit $t\gg l_0^2$ the exponential factor in Eq. (11) can be dropped. The probability distribution $P_l(x_l,t)$ of the leftmost excursion of the right walker (from its initial position) is given by a similar expression. Clearly $l=l_0-x_l-x_r$ is the residual size of the cluster at time t, where $l\leqslant 0$ means the cluster has disappeared. The probability density $p_{l_0}(l,t)$ that the cluster has survived and has size l is therefore given, for $t\gg l_0^2$, by

$$p_{l_0}(l,t)=\int_0^{l_0-l}\frac{dx_l}{\sqrt{\pi t}}\int_0^{l_0-l}\frac{dx_r}{\sqrt{\pi t}}\,\delta(l_0-l-x_l-x_r)$$
$$=\frac{(l_0-l)}{\pi t}. \quad (12)$$

The probability $p_{surv}(l_0,t)$ that the cluster survives until time t is given by

$$p_{surv}(l_0,t)=\int_0^{l_0}dl\,\frac{(l_0-l)}{\pi t}=\frac{l_0^2}{2\pi t}. \quad (13)$$

Immediately we see that the mean interval between surviving clusters grows as t [$=t^{\theta}$, since $\theta(q=\infty)=1$], which is much larger than the mean interval between walkers, which grows as $t^{1/2}$.

The mean length of surviving clusters, of given initial size l_0, in the $t\to\infty$ limit is

$$\langle l\rangle_\infty=\frac{2\pi t}{l_0^2}\int_0^{l_0}dl\,l\,\frac{(l_0-l)}{\pi t}=\frac{1}{3}l_0. \quad (14)$$

If the initial clusters have a distribution of sizes, a straightforward calculation gives $\langle l\rangle_\infty=\langle l_0^3\rangle_0/3\langle l_0^2\rangle_0$, where $\langle\cdots\rangle_0$ indicates an average over this distribution, while the fraction of clusters which survive is $\langle l_0^2\rangle_0/2\pi t$.

The above calculations demonstrate that the mean cluster size approaches a constant at late times and that the scale length for the sizes of the intervals between clusters grows as t, and not as the naive scaling length $t^{1/2}$ associated with the underlying domain wall coarsening. In fact we can calculate the interval-size distribution $n(k,t)$ explicitly for this model. First recall that the surviving clusters are separated, at late times, by many ($\sim t^{1/2}$) walkers. The fate of a given cluster depends only on the nearest walker on either side. The motion of these walkers is uncorrelated with that of the walkers bordering other clusters, because the correlation length for walkers grows only as $t^{1/2}$. Therefore we can assume, in the scaling limit $k\to\infty$, $t\to\infty$, with k/t fixed, that clusters survive *independently* with probability $l_0^2/2\pi t$ (where we have specialized to initial clusters of fixed size l_0). The probability distribution of the interval size k between neighboring clusters is therefore exponential, with mean $\langle k\rangle=2\pi t/l_0^2$, while $n(k,t)=\langle k\rangle^{-2}\exp(-k/\langle k\rangle)$, which has the scaling form (2) with $z=1$. This exponential form is in excellent agreement with simulation data presented in Sec. IV.

Since the initial clusters used in the $q=\infty$ simulations are quite small (2 or 4), the continuum limit is not expected to be quantitatively correct. To conclude this section, therefore, we quote the asymptotic mean cluster size for clusters of arbitrary initial size l_0. The details are given in the Appendix: the result is $\langle l\rangle_\infty=(l_0+2)/3$, which generalizes the continuum result $\langle l\rangle_\infty=l_0/3$.

IV. NUMERICAL SIMULATIONS

Let us recall the zero-temperature coarsening dynamics of the 1D q-state Potts model, starting from a random initial configuration. To maximize the speed of the program, we adopt two-sublattice parallel updating. The zero-temperature dynamics proceeds as follows: in each time step every spin on one of the sublattices changes its color to that of one of its two neighbors with equal probability, the two sublattices being updated alternately. The dynamics of such systems can equivalently be formulated in terms of the motions of the domain walls as a reaction-diffusion model. The domain walls can be considered as random walkers performing independent random walks. Whenever two walkers meet, they annihilate ($A+A\to 0$) with probability $1/(q-1)$ or aggregate ($A+A\to A$), to become a single walker, with probability $(q-2)/(q-1)$. The persistence is the probability that a fixed point in space has not been traversed by any random walker. For $q=2$, the particles only annihilate and hence this is equivalent to the Glauber model, whereas in the $q=\infty$ limit the walkers only aggregate.

This algorithm is, however, restricted to modeling q-state Potts models with $q\geqslant 2$, since $q<2$ generates negative probabilities, while the general result (1) allows any real $q\geqslant 1$. In order to simulate values of q in the interval (1,2) we can map the Potts model (as far as persistence properties are concerned) onto an Ising representation with a fraction $1/q$ of the sites initially pointing up, and initially persistent, while the down spins are nonpersistent. Studying the persistence of the minority (majority) spins then corresponds to the case $q>2$ ($q<2$) [17]. The domain walls between the up and down spins behave as random walkers, annihilating each other on contact with probability 1. As far as the persistence is concerned, the dynamics of the Potts and Ising models are completely equivalent [17]. For $q>2$, results obtained from both types of simulation are entirely consistent.

The numerical simulations are performed on a 1D lattice of size 2×10^6 with periodic boundary conditions. The Ising spins, or Potts variables, occupy the odd sites of the lattice, while the random walkers are restricted to even-numbered sites. Therefore the effective size of the lattice, in terms of Ising or Potts spins, is 10^6. Each walker jumps to an adjacent even-numbered site with equal probability, turning any persistent site it hops over into a nonpersistent site. The positions of all walkers are updated simultaneously at each Monte Carlo step. The initial configurations are chosen to eliminate the possibility of any ''crossover'' of random walkers when they jump. This can be done by placing them only on sites whose positions on the lattice are of the form $4k$, where k is an integer. The walkers then occupy subsets of the sites $4k$ and $4k+2$ alternately.

For the Potts simulations ($q>2$), the lattice is initially completely persistent. The random walkers are periodically

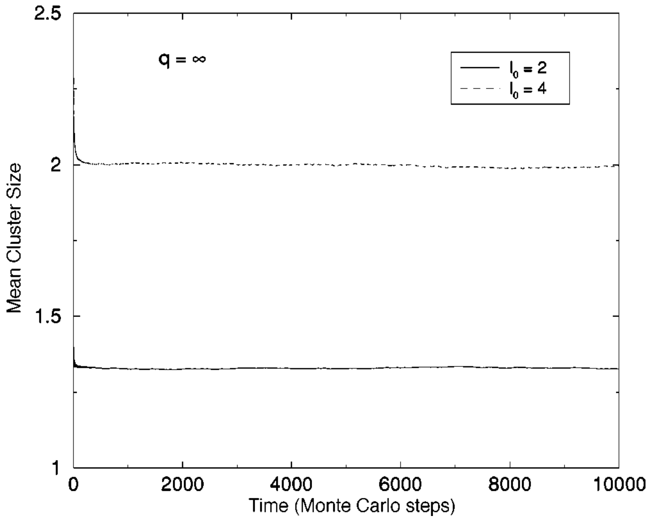

FIG. 2. Mean cluster size as a function of time for $q=\infty$ and initial cluster sizes $l_0=2$ and 4.

laid down, in clusters of uniform initial size l_0. The walkers then execute random walks, aggregating or annihilating according to the prescribed probabilities, $(q-2)/(q-1)$ and $1/(q-1)$, respectively. The simulations are performed for initial cluster sizes of 2 and 4. The values of q chosen are $q=\infty$, 8, and 4, for which the corresponding values of θ are $\theta(\infty)=1$, $\theta(8)\simeq 0.7942$, and $\theta(4)\simeq 0.6315$.

For $q<2$ the Ising representation is employed, in which persistent sites (up spins) are periodically laid down on the lattice, with a fraction $1/q$ of all sites persistent, and the domain walls execute random walks annihilating according to $A+A\to 0$. The value $q=3/2$ was chosen, for which $\theta \simeq 0.2350$. The simulations were performed for initial cluster sizes 4 and 8. The Ising simulations were also run for $q=4$ and 8. The results for scaling functions are consistent with the Potts data, but the statistics are poorer due to the smaller number of clusters generated. For this reason, the data presented in Figs. 3–9 were generated from the Potts runs. All the simulations (Potts and Ising) were run for 10 000 Monte Carlo steps, and the results averaged over 30 independent runs.

The simulations investigate the limiting values of the average sizes of the persistent clusters and the distribution $n(k,t)$ of intervals between persistent clusters with emphasis on its dynamic scaling form. While the first of these lends itself readily to direct measurement, the dynamic scaling form for $n(k,t)$ manifests itself most clearly in the data through a study of the (complement of the) cumulative distribution. We define

$$I(k,t)=\sum_{k'\geqslant k} n(k',t). \tag{15}$$

Inserting the scaling form (2) and converting the sum to an integral in the scaling limit $k\to\infty$, $t\to\infty$, with k/t^z fixed but arbitrary, gives

$$I(k,t)=t^{-2z}\int_k^\infty dk' n(k',t)=t^{-z}\int_{k/t^z}^\infty dx f(x)=t^{-z}g(k/t^z). \tag{16}$$

TABLE I. Average cluster size at late times, $\langle l\rangle_\infty$, for various values of q and various initial cluster sizes l_0, where P or I indicate whether Potts or Ising representations were employed. In all cases the data were averaged over times between 1000 and 9000 Monte Carlo steps.

q	l_0	$\langle l\rangle_\infty$
∞	2(P)	1.330 ± 0.003
	4(P)	1.998 ± 0.006
8	2(P)	1.367 ± 0.002
	2(I)	1.333 ± 0.005
	4(I)	2.018 ± 0.013
4	2(P)	1.3945 ± 0.0007
	4(P)	2.152 ± 0.001
	4(I)	2.089 ± 0.004
1.5	4(I)	2.5880 ± 0.0002
	8(I)	4.572 ± 0.002

We now discuss the numerical results for the $q=\infty$ model. In Fig. 2 we plot the mean cluster size against time for initial cluster sizes of 2 and 4. The numerical results are summarized in Table I. The numerics clearly support the exact results given in Sec. III [and derived in the Appendix as Eq. (A5)], namely, $\langle l\rangle_\infty=(l_0+2)/3$ where l_0 is the initial cluster size. In order to investigate the dynamic scaling form (16) for $q=\infty$, we plot, in Fig. 3, $tI(k,t)$ against k/t for $t=500$, 1000, 2000, 4000, 8000, for initial cluster size 2, where we have used $z=\theta(\infty)=1$ for $q=\infty$. Excellent data collapse occurs in agreement with the dynamic scaling form (16).

In Sec. III we argued that the scaling function $f(x)$ in (2) should be a simple exponential, which implies that $g(x)$ in Eq. (16) is also a simple exponential. To test this prediction, we plot $\ln[tI(k,t)]$ against k/t in Fig. 4. The data lie on the expected straight line except at the latest times, where the deviations from the line are due to statistical noise. Very similar results are obtained for initial cluster size 4, confirming that the scaling function $f(x)$ is independent of the initial walker density.

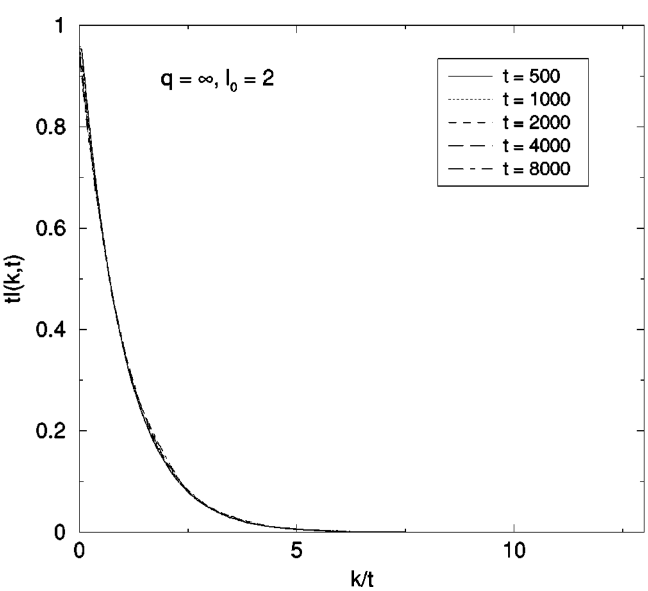

FIG. 3. Scaling plot for $q=\infty$, for initial cluster size $l_0=2$.

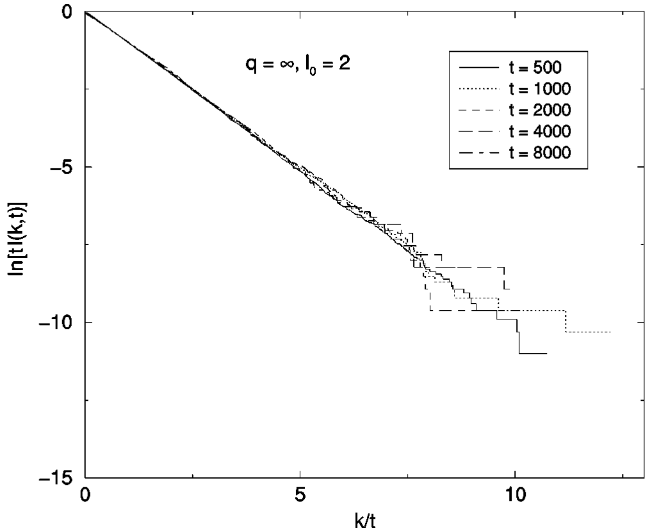

FIG. 4. Same as Fig. 3, but presented as a log-linear plot to reveal the exponential form of the scaling function.

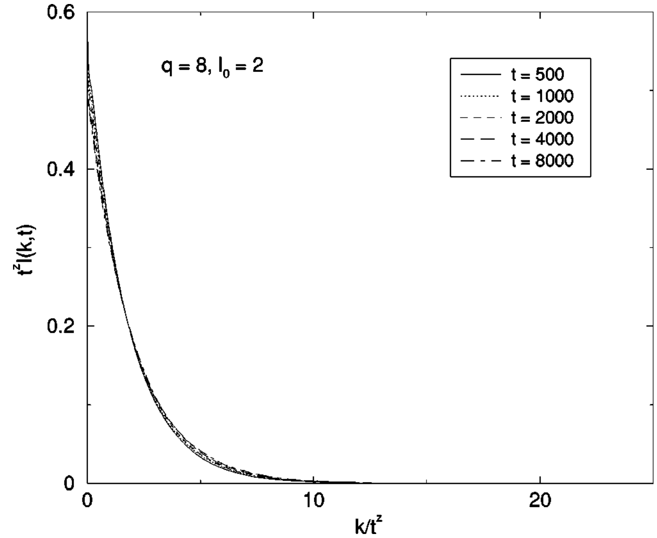

FIG. 6. Scaling plot for $q=8$ and initial cluster size $l_0=2$, with $z=\theta(8)\simeq 0.7942$.

In Fig. 5 the mean cluster size is plotted against time for various finite values of q and initial cluster size l_0. The numerical results indicate that the average cluster size tends to a constant (see Table I) in all cases, as expected on the basis of the heuristic argument in Sec. II. For $q=8$, $\theta(q)\simeq 0.79 > 1/2$ and the dominant length scale is still determined by the mean distance between persistent clusters. We therefore anticipate a scaling form of the kind given by Eq. (16) with $z=\theta(8)$. The scaling plot presented in Fig. 6 shows good data collapse with this value of z, while Fig. 7 indicates that $g(x)$ and, therefore, $f(x)$ are again simple exponentials. The data collapse is not as good as for $q=\infty$, which we attribute to a poorer separation of the length scales L_p and L_w at the time scales available. This problem becomes more acute for $q=4$ (see the discussion below).

For $q=4$, $\theta(q)\simeq 0.63 > 1/2$, so we plot (Fig. 8) $t^z I(k,t)$ against k/t^z with $z=\theta(4)$. This time the scaling collapse is definitely not as good as for $q=\infty$ and $q=8$, reflecting an apparent deviation from the dynamic scaling form (16). We believe that this deviation reflects the proximity of the length scales $L_p \sim t^{0.63}$ and $L_w \sim t^{1/2}$, which are not sufficiently well separated on the time scales achievable in the simulation, and would disappear for asymptotically large times. For example, $L_p/L_w \sim t^{0.1315} \simeq 3.26$ at $t=8000$, while the corresponding ratio is about 14 for $q=8$ and 90 for $q=\infty$. A similar effect may explain the apparent deviations from universality (e.g., an apparent dependence of z on the initial walker density) in the $q=2$ simulations of MR. In this case $\theta=3/8$, and $L_p/L_w \sim t^{-1/8} \sim 0.31$ at $t=12\,000$, the largest time reached by the MR simulations. So in this case also there is not a strong separation of length scales at the times achieved in the simulations (a point recently emphasized by Manoj and Ray [15]).

Figure 9 gives the log-linear plot for $q=4$. Again, the data are consistent with an exponential scaling function, except at small scaling variable where there is a perceptible deviation from a straight line. In this region, however, the

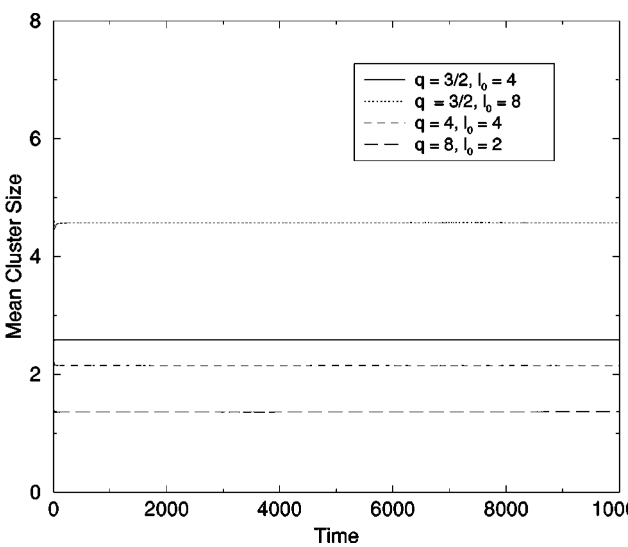

FIG. 5. Mean cluster size as a function of time for various values of q and various initial cluster sizes. The Potts representation was used for $q>2$.

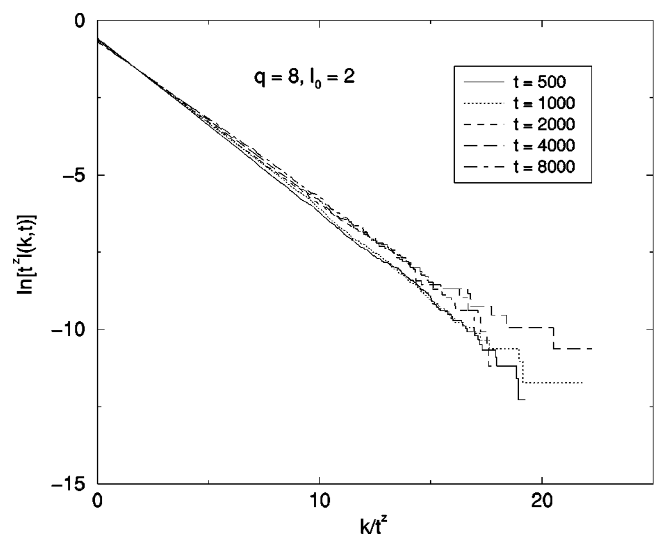

FIG. 7. Same as Fig. 6, but presented as a log-linear plot to reveal the underlying exponential form.

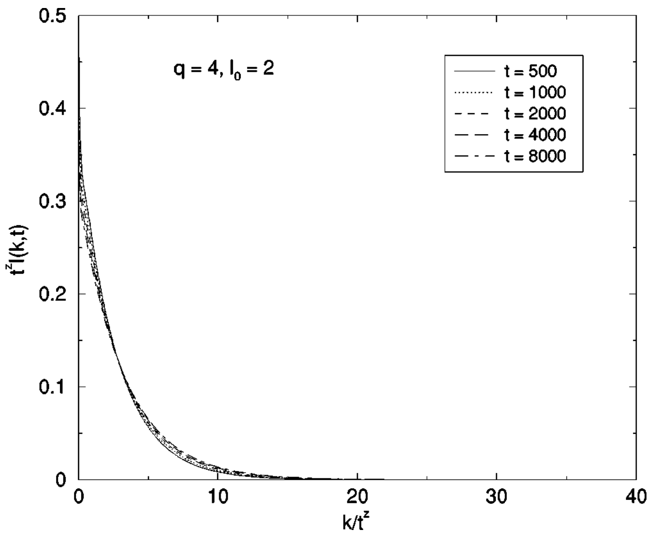

FIG. 8. Same as Fig. 6, but for $q=4$, with $z=\theta(4)\simeq 0.6315$.

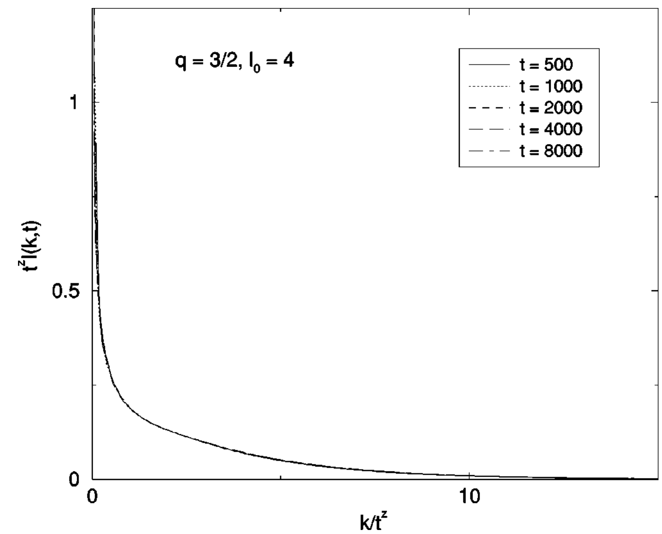

FIG. 10. Same as Fig. 6, but for $q=3/2$, with $z=1/2$.

data also do not scale at all well either (see inset in Fig. 9), which may be another manifestation of the poor separation of length scales: the scaling limit requires $k\to\infty$, $t\to\infty$ with k/t^z fixed. In particular the condition $k \gg L_w \sim t^{1/2}$, which should be satisfied for good scaling, is violated at the small values of k/t^z where the upturn in the data occurs. We will return to this point in the Discussion.

To investigate the spatial distribution of the sites in the regime $\theta<1/2$ we study $q=3/2$, for which $\theta(q)\simeq 0.235$. In Fig. 5 we show that the mean cluster size is a constant for initial cluster sizes of 4 and 8. In the regime $\theta<1/2$ the density of the persistent sites decays more rapidly than that of the walkers. The dominant length scale is therefore no longer given by the persistence length $L_p \sim t^{\theta}$, but is given instead by the mean separation $L_w \sim t^{1/2}$ of the the walkers. Hence we expect asymptotic dynamic scaling of the forms (2) and (16) with $z=1/2$. In Fig. 10 we plot $t^{1/2}I(k,t)$ against $k/t^{1/2}$ for an initial cluster size $l_0=4$. The numerical results give excellent data collapse, supporting the proposed dynamic scaling form. Apart from a change of scale, very similar results are obtained for $l_0=8$, supporting the universality of the scaling function, i.e., the independence of $g(x)$ from the initial walker density. We choose $q=3/2$ rather than $q=2$ in order that the length scales L_p and L_w be reasonably well separated at late times: $L_p/L_w \sim t^{-0.265}\simeq 0.09$ at $t=8000$.

Figure 10 reveals a pronounced upturn at small k, which can be seen more clearly in the log-linear plot of Fig. 11 and its inset, suggesting a divergence for $k\to 0$. This is, in fact, expected from the analysis of Sec. II, which predicts, for $\theta<1/2$, $f(x)\sim x^{-\tau}$ for $x\to 0$, with $\tau=2(1-\theta)$; i.e., τ is in the range $1<\tau<2$. The scaling function $g(x)$ is given by $g(x)=\int_x^\infty dy\, f(y)\sim x^{-(\tau-1)}$ for $x\to 0$, i.e., $g(x)\sim x^{-(1-2\theta)} = x^{-d_f}$, where $d_f=1-2\theta$ is the fractal dimension of the persistent set on scales smaller than L_w [Eq. (8)]. According to this prediction, the product $x^{1-2\theta}g(x)$ should approach a constant at small x. This product is shown in Fig. 12. The small-k divergence has clearly been removed, the function

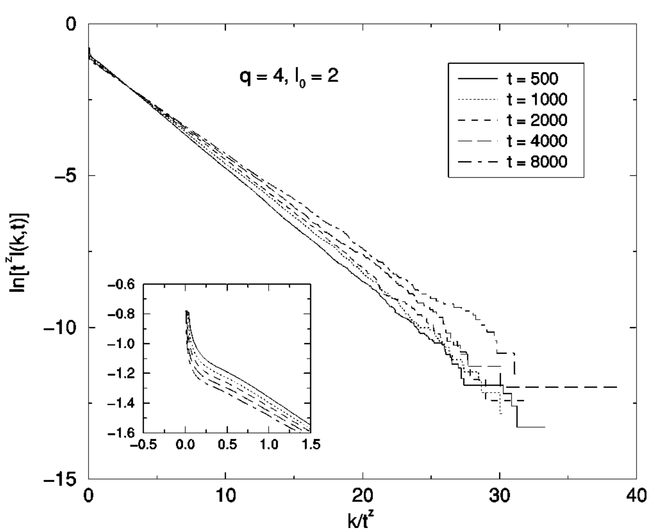

FIG. 9. Same as Fig. 8, but presented as a log-linear plot to test the predicted exponential form of $g(x)$. The inset is an expanded version of the extreme left of the plot.

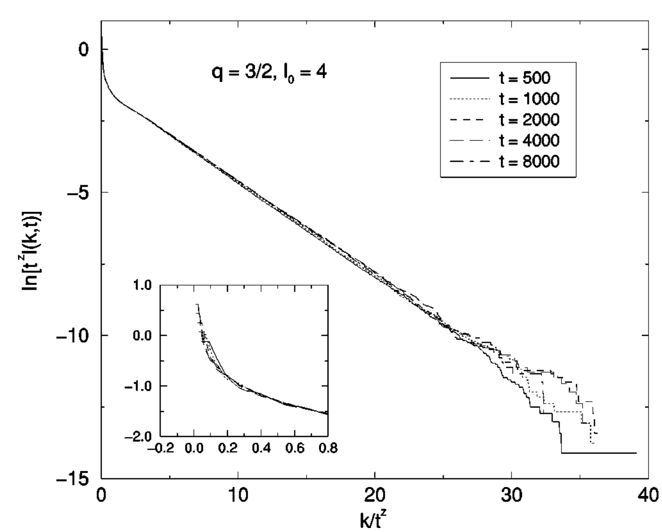

FIG. 11. Same as Fig. 7, but for $q=3/2$ and with $z=1/2$. The inset shows an expanded version of the extreme left of the plot and suggests a singularity for small $x=k/t^{1/2}$.

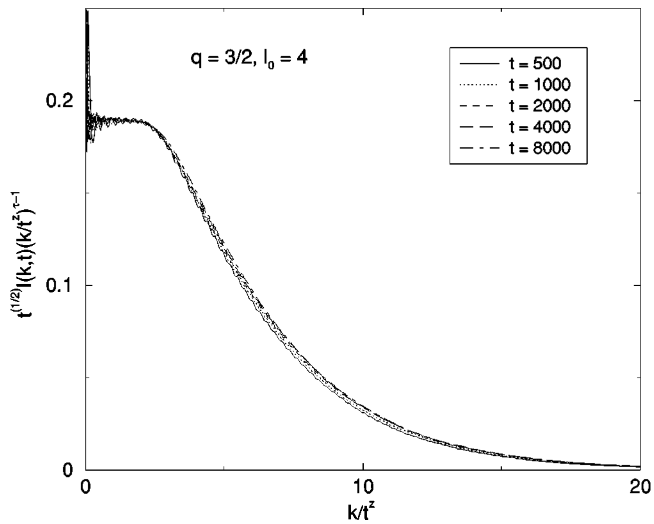

FIG. 12. Same data as Fig. 10, but replotted with ordinate $t^{1/2}[k/t^{1/2}]^{1-2\theta}I(k,t)$ to show the small-x behavior ($x=k/t^{1/2}$) more clearly.

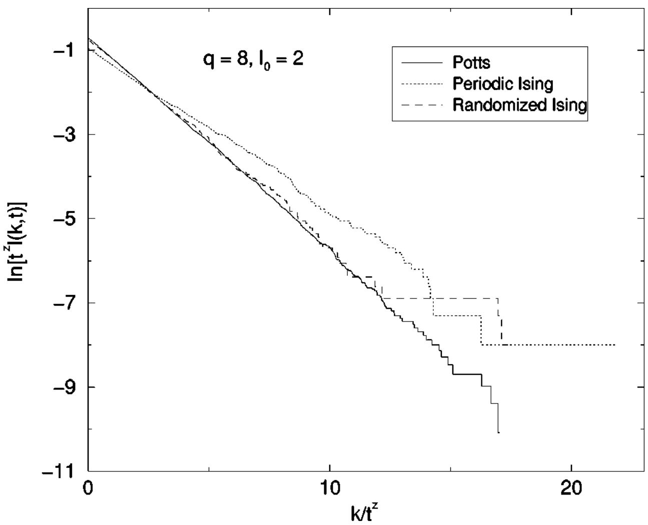

FIG. 13. Scaling function for $q=8$, showing equivalence of reaction-diffusion (''Potts'') and Ising (''randomized Ising'') representations, with $z=\theta(8)$, where the axes are $qt^zI(k,t)$ and k/qt^z for the Ising data. The Ising data for periodic initial conditions (''periodic Ising'') has the same functional form but a different amplitude.

approaching a constant at small x as predicted (the erratic behavior at very small x is due to statistical noise).

For $q=3/2$ all of the numerical simulations have been performed using the Ising representation, with periodic initial conditions in which clusters of l_0 ''up spins'' are placed at uniform intervals in a ''down spin'' background, and occupy a fraction $1/q$ of the sites. For $q>2$, the Potts representation has mainly been used, but we expect all universal properties, such as exponents and scaling functions, to be the same for the two representations. This expectation is confirmed by the results for $\theta(q)$ and the scaling function $g(x)$. In order to obtain a *precise* correspondence between the Ising and the Potts representations, however, it is necessary to run the Ising simulations with random initial conditions, rather than periodic initial conditions which we have mostly employed, and to scale the axes appropriately with q to account for the different interval sizes in the two cases. This is because the reaction-diffusion representation of the Potts model, in which walkers annihilate or coalesce with probabilities which depend on q, is only an exact representation if the Potts states occur in a completely random sequence. This is not true for the periodic initial conditions employed in the Ising representation, whose correlations spoil the exact correspondence between Ising and Potts simulations. Our expectation, then, is that the Ising representation with random and periodic initial conditions should give the same *scaling functions* as the reaction-diffusion implementation of the Potts model, but nonuniversal amplitudes will be different for the periodic initial conditions.

In order to test the exact equivalence between random Ising and Potts simulations we plot, in Fig. 13, the results for the $q=8$ model using the Potts (reaction-diffusion) method. The lattice is initially all persistent and the initial cluster size is $l_0=2$. This corresponds to a sequence of pairs of Potts states in which the members of a pair are in same state, but there are no correlations between pairs (except that neighboring pairs must be in different states). The random initial conditions for the Ising simulation are generated from the Potts initial state by setting all pairs in Potts state ''1'' (say) to be up spins and all other pairs to be down spins. In the Potts simulations we keep track of all persistent sites, while in the Ising simulation we track the persistence of the spins initially ''up.'' The interval sizes (between persistent clusters) are then naturally larger by a factor q in the Ising simulations relative to the Potts simulations. Figure 13 shows both scaling functions in log-linear form, for $t=10^4$, with this factor of q scaled out [i.e., $qt^zI(k,t)$ is plotted against k/qt^z for these data]. The data overlap almost perfectly. The Ising data for a periodic initial condition are consistent with the same scaling function (exponential), but with a different amplitude.

V. DISCUSSION AND SUMMARY

In this paper we have investigated the nature of persistent structures in the coarsening dynamics of 1D Potts models. A central concept has been the existence of two characteristic length scales, the mean separation $L_w \sim t^{1/2}$ of domain walls (or ''walkers'') and the ''persistence length,'' $L_p \sim t^{\theta}$, which measures the mean separation of persistent sites or clusters. The focus of our attention has been the distribution function $n(k,t)$ for the number of intervals (between clusters of persistent sites) of length k. This distribution has the scaling form (2), with characteristic length scale t^z, which is the *larger* of L_w and L_p, i.e., $z=\max(1/2,\theta)$. Within the 1D Potts model both regimes $\theta<1/2$ and $\theta>1/2$ can be accessed by varying the number of Potts states q. From the general result, Eq. (1), we see that $\theta(q)$ is a monotonically increasing function with $q_c=2.705\,28\ldots$ marking the boundary between the two regimes.

The regime $\theta>1/2$ is conceptually simpler. The $q=\infty$ limit can been solved exactly (Sec. III), and results for the long-time limit of the mean cluster size obtained. The fact that the locations of the surviving clusters are statistically independent in the scaling regime leads to the result that the scaling function $f(x)$ in Eq. (2) is a simple exponential. We have argued that the same result should hold for all θ

$>1/2$, since in this regime there are, on average, many walkers between each pair of persistent clusters. The *relevant* walkers for this argument (those which are turning persistent sites in neighboring clusters nonpersistent) are uncorrelated since their separations, of order L_p, are much larger than the typical separation L_w of neighboring walkers. On this basis we expect the asymptotic scaling function $f(x)$ to be exponential for all $\theta > 1/2$ (i.e., $q > q_c$). The data for $q=8$ (Fig. 7) and $q=4$ (Fig. 9) are consistent with an exponential form for $g(x) = \int_x^\infty f(x)$, although the scaling is not perfect and there is a small upturn in the scaling function at small scaling variable for $q=4$. We attribute these features to an imperfect separation of length scales on the time scales achieved in the simulations and conjecture that the true scaling function is exponential for all $\theta > 1/2$.

The case $\theta < 1/2$ is more tricky. In this case the persistent clusters outnumber the walkers. The scaling function $f(x)$ is clearly *not* a simple exponential, though it seems (Fig. 11) to have an exponential tail. There is a small-argument singularity of the form $f(x) \sim x^{-\tau}$, with $\tau = 2(1-\theta)$. This is related to the fractal dimension $d_f = 1 - 2\theta$ of the persistent sites: $d_f = \tau - 1$. Note that the borderline $\theta = 1/2$ between the two regimes occurs at $d_f = 0$. The existence of a small-x singularity for $\theta < 1/2$ raises the possibility of an alternative scenario for $\theta > 1/2$, in which the $x^{-\tau}$ singularity, with $\tau = 2(1-\theta)$, persists for $\theta > 1/2$, where it becomes an integrable singularity. The small-x singularity in $g(x)$ would then take the form of a cusp: $g(x) = g(0) - Ax^{2\theta - 1} + \cdots$. We have not been able to rule out this scenario from the data, but think it unlikely for the reasons given elsewhere in this paper. Further insight could be obtained if it were possible to perform an expansion around the soluble $q = \infty$ limit to first order in $1/q$, but we leave this as a challenge for the future.

We conclude by discussing briefly the possibility of the existence of these two qualitatively different regimes in spatial dimension $d \geq 2$. First we generalize the result (8), relating d_f and θ, to any dimension. Starting from Eq. (7), the result $F(x) \sim x^{-2\theta}$ follows generally, and the number of persistent sites within a distance R of a given site is estimated as $\int_0^R r^{d-1} dr\, r^{-2\theta} \sim R^{d_f}$, where $d_f = d - 2\theta$. Generalizing still further, if the coarsening exponent is ϕ, rather than 1/2, the result is

$$d_f = d - \theta/\phi \tag{17}$$

(the $d=2$ version of this result is given in [16], based on the same reasoning). Clearly this result requires $\theta \leq d\phi$, since d_f cannot be negative. If this inequality is violated, as in the 1D Potts model with $\theta > 1/2$, the persistent sites no longer have a fractal structure but become pointlike objects, with mean density $\sim t^{-\theta} \ll t^{-d\phi} \sim L_c^{-d}$, where $L_c \sim t^\phi$ is the coarsening length scale. Cases where $\theta < d\phi$ are easy to find: e.g., in the coarsening of the 2D Ising model [4], the 2D diffusion equation [5], or the time-dependent Ginzburg-Landau equation in 2D. These all exhibit fractal persistent structures with the expected fractal dimension [16,18]. It would be interesting to look for examples, in addition to the 1D Potts model, where one can have $\theta > d\phi$.

Note added in proof. Tam [19] have recently investigated an experimental system, namely a coarsening two-dimensional soap froth, for which the inequality, discussed at the end of Sec. V, holds. For this system $d=2$, $\phi = \frac{1}{2}$, and $\theta \simeq 1.3$. They find that, in the scaling regime, a decreasing fraction of cells have persistent cores, and that the average core size is constant. The mean spacing between such cells increases as $L_p \sim t^{\theta/2}$ i.e., it is larger than the coarsening length scale $t^{1/2}$. All this is very similar to the $d=1$ Potts model results for $\theta > \frac{1}{2}$, and is in accord with the general scenario outlined in this paper. We thank Andrew Rutenberg and Ben Vollmayr-Lee for bringing this work to our attention.

ACKNOWLEDGMENT

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APPENDIX

In this appendix we compute the probability that a cluster of initial size l_0 survives to time t and the mean size of surviving clusters for the process $A + A \to A$.

First note that initially there is a random walker at each end of the domain. At each time step the walkers move independently left or right with probability 1/2, so we can treat each walker independently. Consider, therefore, a single walker moving at discrete time steps on a discrete 1D lattice, starting, at time $t=0$, at position r. Let the "origin" be the point $r=0$, and let $P_r(t)$ be the probability that the walker has not yet reached the origin at time t. Clearly,

$$P_1(t) = \frac{1}{2} P_2(t-1),$$

$$P_r(t) = \frac{1}{2}[P_{r-1}(t-1) + P_{r+1}(t-1)], \quad r \geq 2. \tag{A1}$$

We are interested only in the limit $t \to \infty$. In this limit, we know from standard random walk theory that every $P_r(t)$ decays like $t^{-1/2}$, with an r-dependent amplitude. To leading order in $t^{-1/2}$, therefore, the t dependence drops out of Eqs. (A1), which then become equations for the amplitudes. By inspection, the solution in this regime is

$$P_r(t) = r P_1(t). \tag{A2}$$

Now consider a walker starting immediately to the right of a cluster of l_0 persistent sites. The probability that after t steps the walker has jumped over exactly r of these (making them nonpersistent) is $P_{r+1}(t) - P_r(t) = P_1(t)$, where the final result follows from Eq. (A2), to leading order for large t. The same result holds for a walker starting immediately to the left of the cluster. The probability that l sites remain persistent after time t is $P_1(t)^2$ times the number of ways of partitioning the cluster of length l_0 into three sections, with the central section of length l (and zero lengths are allowed for the outer sections). This number is $l_0 - l + 1$. So the probability of the cluster surviving and having length l is

$$p_{l_0}(l,t)=(l_0-l+1)P_1(t)^2, \tag{A3}$$

a generalization of Eq. (12) to the discrete system.

The survival probability of the cluster is

$$p_{surv}(t)=\sum_{l=1}^{l_0} p_{l_0}(l,t)=\frac{1}{2}l_0(l_0+1)P_1(t)^2, \tag{A4}$$

while the mean cluster size is

$$\langle l\rangle_\infty=\frac{\sum_{l=1}^{l_0} l p_{l_0}(l,t)}{\sum_{l=1}^{l_0} p_{l_0}(l,t)}=\frac{(l_0+2)}{3}. \tag{A5}$$

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